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On the Relation between Linear n-Widths and Approximation Numbers

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We show that for a compact absolutely convex subset of a normed space the linear n-width obtained with the help of bounded linear rank n operators, the one obtained by arbitrary linear rank n operators, and the corresponing approximation number of the associated embedding, all coincide. This is achieved by a version of the principle of local reflexivity for spaces connected with embedding operators. We also give a counterexample showing that for relatively compact absolutely convex sets equality does not hold any longer and the discrepancy can be of the (maximally possible) order $n^{1/2}$. © 1989 Academic Press, Inc.

INTRODUCTION AND NOTATION

There are three frequently used concepts for describing linear approximability: Given an absolutely convex (i.e., convex and balanced) set A in a normed linear space X, the linear n-width is defined as

$$\lambda_n(A, X) = \inf_{\substack{S \in L(X) \\ \operatorname{rk} S \leqslant n}} \sup_{x \in A} \|x - Sx\|,$$

where $L(X)$ is the set of all bounded linear operators on X (see Tichomirov [17] and Pinkus [15]). Some authors (see, e.g., Helfrich [3] for a detailed study) drop the boundedness requirement, which leads to the second concept—let us call it the algebraic linear n-width,

$$\lambda_n^{\#}(A, X) = \inf_{\substack{S \subset L^{\#}(X) \\ \operatorname{rk} S \leqslant n}} \sup_{x \in A} \|x - Sx\|,$$

where $L^{\#}(X)$ denotes the set of all linear (not necessarily bounded)

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operators on X. Finally, for a bounded linear operator $T \in L(Y, X)$ from a normed space Y to X the nth approximation number is defined as

$$a_n(T) = \inf_{\substack{S \in L(Y, X) \\ \operatorname{rk} S < n}} \|T - S\|$$

(see Pietsch [14]). To connect the last concept with sets in normed spaces, let $A \subset X$ be absolutely convex and bounded. Then consider the space X_A generated by A; that is, X_A constitutes the linear span of A, endowed with the norm

$$\|x\|_A = \inf\{\lambda > 0 : x \in \lambda A\} \qquad (x \in \operatorname{span} A).$$

Let $J_A : X_A \to X$ be the identical embedding. Thus the approximation numbers of J_A reflect linear approximability of A, as well. This suggests the problem of the relation between all three concepts, or more precisely, between $\lambda_n(A, X)$, $\lambda_n^{\#}(A, X)$, and $a_{n+1}(J_A)$ (the shift of index is appropriate due to the different rank restrictions customary in the definitions of n-widths and approximation numbers).

The main result of this paper is the coincidence of $\lambda_n(A, X)$ and $\lambda_n^{\#}(A, X)$ for all sets A of the form $K + E$, where K is a compact absolutely convex subset and is E a finite-dimensional subspace of X. For A compact and absolutely convex, all three quantities $\lambda_n(A, X)$, $\lambda_n^{\#}(A, X)$, and $a_{n+1}(J_A)$ coincide. The key step in the proof is a version of the principle of local reflexivity for spaces related to embedding operators. We obtain parallel results for Gelfand n-widths and numbers, as well as for projectional n-widths, by the help of reductions to the case of linear n-widths. An example shows that compactness cannot be replaced by relative compactness. In this example the deviation of λ_n from $\lambda_n^{\#}$ is of order $n^{1/2}$.

The paper is organized as follows. Section 1 contains some simple general relations between n-widths and s-numbers. In Section 2 we state the principle of local reflexivity due to Lindenstrauss and Rosenthal [10] and derive from it the needed version related to embedding operators. The main results on the coincidence of n-widths and s-numbers are contained in Section 3. Section 4 is devoted to the counterexample concerning relatively compact sets.

Let us now give some more notation. The normed linear spaces under consideration are assumed to be defined either all over the field of reals, or all over the field of complex numbers. Given a normed space X, by X^* we denote the space of all continuous linear functionals on X and by B_X the closed unit ball of X. The "continuous" and "algebraic" linear n-widths have been defined above. Next let us recall the Gelfand and Kolmogorov counterparts. The Gelfand n-width is defined as

$$d^n(A, X) = \inf_{\substack{N \in S(X) \\ \operatorname{codim} N \leqslant n}} \sup_{x \in N \cap A} \|x\|,$$

where $S(X)$ denotes the set of closed subspaces of X. (Note that for all occurring n-widths we admit the value $+\infty$, since we do not require boundedness of A.) The algebraic Gelfand n-width is given by

$$d^{n,\#}(A, X) = \inf_{\substack{N \in S^{\#}(X) \\ \operatorname{codim} N \leqslant n}} \sup_{x \in N \cap A} \|x\|,$$

where $S^{\#}(X)$ is the set of all (not necessarily closed) subspaces of X. Given an operator $T \in L(Y, X)$, the nth Gelfand number is defined by

$$c_n(T) = \inf_{\substack{N \in S(Y) \\ \operatorname{codim} N < n}} \sup_{y \in B_Y \cap N} \|Ty\|.$$

Finally, the Kolmogorov n-width is given by

$$d_n(A, X) = \inf_{\substack{E \in S(X) \\ \dim E \leqslant n}} \sup_{x \in A} \inf_{z \in E} \|x - z\|$$

(clearly, in this case an algebraic version would not give anything new), and the nth Kolmogorov number of $T \in L(Y, X)$ is defined as

$$d_n(T) = \inf_{\substack{E \in S(X) \\ \dim E < n}} \sup_{y \in B_Y} \inf_{z \in E} \|Ty - z\|.$$

Obviously, $d_n(TB_Y, X) = d_{n+1}(T)$. The theory of n-widths is developed in Tichomirov [17] and Pinkus [15]; for the algebraic versions see Helfrich [3]. The theory of s-numbers, for which approximation, Kolmogorov, and Gelfand numbers are examples, can be found in Pietsch [14]. For notation and results concerning Banach spaces we refer the reader to Lindenstrauss and Tzafriri [1].

1. n-Widths and s-Numbers

First we shall establish a connection between the linear and Gelfand n-widths for Banach spaces X with the metric extension property [14, C.3], which means that for each normed space Y and each bounded linear operator $T \in L(Z, X)$ from a subspace Z of Y to X there is an extension

$\tilde{T} \in L(Y, X)$ of T to Y with $\|\tilde{T}\| = \|T\|$. For example, for each set Γ the space $l_\infty(\Gamma)$ has the metric extension property.

1.1. LEMMA. *Let X be a Banach space with the metric extension property and let $A \subseteq X$ be an absolutely convex set. Then for all n,*

(i) $d^n(A, X) = \lambda_n(A, X)$, *and*

(ii) $d^{n,\#}(A, X) = \lambda_n^\#(A, X)$.

Statement (i) for bounded A is due to Ismagilov [5, Th. 1 and Cor. to Th. 2]. Since we have included infinite values, let us agree that in the statement above as well as in 1.2 we mean that either both quantities involved are finite and satisfy the relation, or both are infinite.

Proof. First we shall prove (ii) and then mention how (i) follows. Let $d = d^{n,\#}(A, X) < \infty$. Let $\varepsilon > 0$ and choose $N \in S^\#(X)$ with $\operatorname{codim} N \leqslant n$ such that

$$A \cap N \subseteq (d + \varepsilon) B_X. \tag{1}$$

Let $X_A = \operatorname{span} A$ and let p_A be the seminorm on X_A defined by A, i.e., $p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}$ for $x \in X_A$. Define

$$Y = X_A / \{x \in X_A : p_A(x) = 0\}$$

and denote the quotient map from X_A to Y by Q. So p_A is a norm on Y. Let

$$Z = Q(X_A \cap N).$$

It follows from (1) that Q is injective on $X_A \cap N$. Therefore we can define a linear operator T from Z to X by setting for $z \in Z$,

$$Tz = Q^{-1} z.$$

By (1),

$$T \in L(Z, X) \qquad \text{and} \qquad \|T\| \leqslant d + \varepsilon.$$

Using the metric extension property of X, we obtain a $\tilde{T} \in L(Y, X)$ with $\tilde{T}|_Z = T$ and

$$\|\tilde{T}\| \leqslant d + \varepsilon.$$

Now we define a linear operator from X_A to X by setting $S_0 = J_A - \tilde{T}Q$, where J_A is the identical embedding of X_A into X. It follows that

$$X_A \cap N \subseteq \operatorname{Ker} S_0.$$

Clearly we can find an algebraic extension S of S_0 to all of X such that

$$N \subseteq \operatorname{Ker} S. \tag{2}$$

Hence rk $S \leqslant n$ and, since S is an extension of S_0, we have for $x \in A$

$$\|x - Sx\| = \|\tilde{T}Qx\| \leqslant \|\tilde{T}\| \; p_A(Qx) \leqslant d + \varepsilon.$$

This proves (ii). Note that (i) immediately follows from this proof: If N is closed, then (2) implies that S is continuous.

Now we shall study the relation between continuous and algebraic n-widths in the case of general A and X.

1.2. PROPOSITION. *Let X be a normed space and let $A \subseteq X$ be an absolutely convex subset of X. Then for all n,*

(i) $\lambda_n^{\#}(A, X) \leqslant \lambda_n(A, X) \leqslant (n^{1/2} + 1)\lambda_n^{\#}(A, X)$, *and*

(ii) $d^{n,\#}(A, X) \leqslant d^n(A, X) \leqslant (n^{1/2} + 1) d^{n,\#}(A, X)$.

Proof. (i) (compare [4, Th. 1.1]). The left-hand inequality is trivial. Assume that $\lambda_n^{\#}(A, X)$ is finite. Clearly, $d_n(A, X) \leqslant \lambda_n^{\#}(A, X)$. So let $\varepsilon > 0$, and let $F \subset X$ be a subspace of dimension at most n such that

$$\sup_{x \in A} \inf_{y \in F} \|x - y\| \leqslant d_n(A, X) + \varepsilon.$$

Let $P: X \to F$ be a continuous projection with $\|P\| \leqslant n^{1/2}$ [7]. Then for $x \in A$,

$$\begin{aligned} \|x - Px\| &\leqslant (n^{1/2} + 1) \inf_{y \in F} \|x - y\| \\ &\leqslant (n^{1/2} + 1)(d_n(A, X) + \varepsilon) \\ &\leqslant (n^{1/2} + 1)(\lambda_n^{\#}(A, X) + \varepsilon). \end{aligned}$$

To prove (ii), we embed X isometrically into $Z = l_\infty(B_{X^*})$ by mapping x to the function $\langle x, x^* \rangle$ $(x^* \in B_{X^*})$. Then

$$d^{n,\#}(A, X) = d^{n,\#}(A, Z)$$

and, by the Hahn–Banach Theorem, also

$$d^n(A, X) = d^n(A, Z).$$

Now (ii) is a consequence of (i) and Lemma 1.1.

In Section 4 it is shown that the estimates above are asymptotically sharp. The next result shows that for embedding operators, s-numbers and algebraic n-widths coincide. This leaves us with the essential part of the problem—the relation between continuous and algebraic widths.

1.3. PROPOSITION. *Let X and Y be normed linear spaces and let $J \in L(Y, X)$ be an injection. Then*

(i) $a_{n+1}(J) = \lambda_n^{\#}(JB_Y, X)$, *and*

(ii) $c_{n+1}(J) = d^{n,\#}(JB_Y, X)$.

Proof. (i) Let $\varepsilon > 0$ and let $S \in L^{\#}(X)$ with rk $S \leqslant n$ and

$$\|Jy - SJy\| \leqslant \lambda_n^{\#}(JB_Y, X) + \varepsilon \qquad (y \in B_Y).$$

This shows that $SJ \in L(Y, X)$, which implies $a_{n+1}(J) \leqslant \lambda_n^{\#}(JB_Y, X)$. To see the converse inequality, it is sufficient to note that each operator $S \in L(Y, X)$ induces a linear (not necessarily continuous) operator from JY to X and can therefore be extended algebraically to all of X. To reduce (ii) to (i), we proceed as in the proof of 1.2(ii) using, in addition, that $c_n(T) = a_n(T)$ for each Banach space X with the metric extension property and each $T \in L(Y, X)$ [14, 11.5.3].

Proposition 1.3 allows us to clarify where a connection between λ_n and $\lambda_n^{\#}$ could come from. Let us briefly comment on this. Let $J \in L(Y, X)$ be any injection. By 1.3 we have

$$\lambda_n^{\#}(JB_Y, X) = \inf_{\substack{y_1^*, \dots, y_n^* \in Y^* \\ x_1, \dots, x_n \in X}} \sup_{y \in B_Y} \left\| Jy - \sum_{k=1}^{n} \langle y, y_k^* \rangle x_k \right\| \tag{3}$$

while

$$\lambda_n(JB_Y, X) = \inf_{\substack{x_1^*, \dots, x_n^* \in X^* \\ x_1, \dots, x_n \in X}} \sup_{y \in B_Y} \left\| Jy - \sum_{k=1}^{n} \langle Jy, x_k^* \rangle x_k \right\|$$

and thus, with $Z = J^*X^*$,

$$\lambda_n(JB_Y, X) = \inf_{\substack{z_1, \dots, z_n \in Z \\ x_1, \dots, x_n \in X}} \sup_{y \in B_Y} \left\| Jy - \sum_{k=1}^{n} \langle y, z_k \rangle x_k \right\|. \tag{4}$$

Hence we will be able to handle the relation between algebraic and continuous n-widths if we are able to handle the relation between Y^* and its subspace Z. The next section is devoted to this topic.

Remarks. 1. Using the same arguments as in the proof of 1.2 and 1.3, one can check that for arbitrary $T \in L(Y, X)$ the following hold,

$$a_{n+1}(T) \leqslant \lambda_n^{\#}(TB_Y, X) \leqslant \lambda_n(TB_Y, X) \leqslant (n^{1/2}+1)\, a_{n+1}(T)$$

and

$$c_{n+1}(T) \leqslant d^{n,\#}(TB_Y, X) \leqslant d^n(TB_Y, X) \leqslant (n^{1/2}+1)\, c_{n+1}(T).$$

2. For operators other than injections no relation like 1.3 can be expected, even in finite dimension. To see this let $X = l_\infty^{2n}$, let B_2^{2n} be the unit ball of l_2^{2n}, and let $Y = l_1^{m}$, where m is so large that there is a surjection $Q \in L(l_1^m, l_2^{2n})$ with $\|Q\| \leqslant 2$ and $QB_Y \supseteq B_2^{2n}$. Let $I_{2,\infty}^{2n}: l_2^{2n} \to l_\infty^{2n}$ be the identity and set $T = I_{2,\infty}^{2n} Q$. Since X and Y are finite-dimensional, algebraic and continuous n-widths clearly coincide. Furthermore, by 1.3 and [14, 11.11.8 and 11.7.4]

$$\lambda_n^{\#}(TB_Y, X) \geqslant \lambda_n^{\#}(B_2^{2n}, l_\infty^{2n}) = a_{n+1}(I_{2,\infty}^{2n}) = 2^{-1/2},$$

while, in view of [14, C. 3 and 11.6.3],

$$a_{n+1}(T) = d_{n+1}(T) \leqslant 2 d_{n+1}(I_{2,\infty}^{2n}) = 2 d_n(B_2^{2n}, l_\infty^{2n}) \leqslant \gamma n^{-1/2},$$

where γ denotes an absolute constant. The right-hand inequality is Kashin's result [8, Th. 1]; see also Lemma 4.1 below. Since l_∞^{2n} has the metric extension property, the same example can also serve for 1.3(ii).

2. A Version of Local Reflexivity

Let us first recall the principle of local reflexivity due to Lindenstrauss and Rosenthal [10, Th. 31] and Johnson *et al.* [6]. Note that we identify a normed space Z with its canonical image in its second dual Z^{**}.

2.1. *Principle of local reflexivity.* Let Z be a normed space, let $G \subset Z^{**}$ and $F \subset Z^*$ be finite-dimensional subspaces, and let $\varepsilon > 0$. Then there exists a linear operator $U: G \to Z$ such that

(i) $(1-\varepsilon)\|z^{**}\| \leqslant \|Uz^{**}\| \leqslant (1+\varepsilon)\|z^{**}\|$ for all $z^{**} \in G$,

(ii) $Uz = z$ for all $z \in G \cap Z$, and

(iii) $\langle Uz^{**}, z^* \rangle = \langle z^*, z^{**} \rangle$ for all $z^* \in F$, $z^{**} \in G$.

We shall apply this to the situation considered in Section 1. It turns out that for an injection $J \in L(Y, X)$ there is a "local reflexivity relation" between Y^* and $Z = J^*X^*$ provided the image of the unit ball JB_Y is weakly compact.

2.2. THEOREM. *Let X and Y be normed spaces and let $J \in L(Y, X)$ be an injection such that JB_Y is weakly compact. Denote $Z = J^*X^*$. Let G be a finite-dimensional subspace of Y^*, F a finite-dimensional subspace of Y, and let $\varepsilon > 0$. Then there is a linear operator $V: G \to Z$ such that*

(i) $(1-\varepsilon)\,\|y^*\| \leqslant \|Vy^*\| \leqslant (1+\varepsilon)\,\|y^*\|$ *for all* $y^* \in G$,

(ii) $Vz = z$ *for all* $z \in G \cap Z$, *and*

(iii) $\langle y, Vy^* \rangle = \langle y, y^* \rangle$ *for all* $y \in F$, $y^* \in G$.

Proof. We shall reduce this to 2.1 by a suitable embedding of Y^* into Z^{**}. Let $I_Z: Z \to Y^*$ be the identical embedding, and let $R: X^* \to Z$ be J^*, considered as an operator to Z; i.e., we have

$$J^* = I_Z R.$$

Consequently,

$$J^{**} = R^* I_Z^*.$$

It follows that

$$R^* B_{Z^*} = J^{**} B_{Y^{**}} = JB_Y, \tag{5}$$

where the last equality is a consequence of the weak compactness of JB_Y. Hence $J^{-1}R^*$ is a bounded linear operator from Z^* to Y. We define $S: Y^* \to Z^{**}$ as

$$S = (J^{-1}R^*)^*.$$

It follows from (5) that $B_Y = J^{-1}R^*B_{Z^*}$ which implies that S is an isometry. Next we show that S is the identity on Z. So let $z \in Z$, $z = J^*x^*$ for some $x^* \in X^*$. Then

$$\begin{aligned}\langle z^*, Sz \rangle &= \langle J^{-1}R^*z^*, J^*x^* \rangle \\ &= \langle R^*z^*, x^* \rangle \\ &= \langle Rx^*, z^* \rangle \\ &= \langle J^*x^*, z^* \rangle \\ &= \langle z, z^* \rangle.\end{aligned}$$

Finally, we check that S preserves the values at elements of Y. More precisely (recalling that $I_Z^*: Y^{**} \to Z^*$), we have

$$\langle I_Z^*\, y, Sy^* \rangle = \langle y, y^* \rangle \tag{6}$$

for all $y \in Y$, $y^* \in Y^*$. Indeed,

$$\langle I_Z^* y, Sy^* \rangle = \langle J^{-1} R^* I_Z^* y, y^* \rangle = \langle J^{-1} J^{**} y, y^* \rangle = \langle y, y^* \rangle.$$

Now we are ready to apply the principle of local reflexivity: There is a linear operator $U: SG \to Z$ with

$$(1-\varepsilon) \|Sy^*\| \leqslant \|USy^*\| \leqslant (1+\varepsilon) \|Sy^*\| \qquad (y^* \in G), \tag{7}$$

$$Uz = z \qquad (z \in SG \cap Z) \tag{8}$$

and

$$\langle USy^*, I_Z^* y \rangle = \langle I_Z^* y, Sy^* \rangle \qquad (y^* \in G, y \in F). \tag{9}$$

We set $V = US$. Then (7) gives (i), since S is an isometry. Moreover, S is the identity on Z; thus $S(G \cap Z) \subseteq SG \cap Z$ and (8) gives (ii). Finally, (9) combined with (6) yields for $y \in F$, $y^* \in G$,

$$\langle y, Vy^* \rangle = \langle y, I_Z Vy^* \rangle = \langle Vy^*, I_Z^* y \rangle = \langle I_Z^* y, Sy^* \rangle = \langle y, y^* \rangle,$$

which is (iii), accomplishing the proof.

Remarks. 1. For separable Z, one can recover the principle of local reflexivity from 2.2. Let $(z_n) \subset Z$ be a sequence with dense linear span and $\|z_n\| \to 0$. Let $Y = Z^*$, $X = c_0$, and $J: Y \to X$ be defined by

$$Jz^* = (\langle z_n, z^* \rangle).$$

Theorem 2.2 gives the local reflexivity for $Z_0 = J^* X^*$ and, since Z_0 is dense in Z, a perturbation argument shows 2.1.

2. The assumption that JB_Y is closed is essential. Kürsten [9, Sect. 9, example 1] showed that there is a sequence (u_n) in l_∞ which is isometrically equivalent to the unit vector basis of l_1 and whose span is w^*-dense in l_∞. Thus, we let $Y = l_1$, $X = c_0$, and $J: Y \to X$ be defined as

$$Jy = \left(\frac{1}{n} \langle y, u_n \rangle\right).$$

By the w^*-density of $J^* X^*$, J is an injection. Moreover, JB_Y is relatively compact in X. However, (i) of 2.2 cannot hold, since the closure of $Z = J^* X^*$ is isometric to l_1, thus of cotype 2, while $Y^* = l_\infty$ is not [11, II, p. 73].

3. Equality of n-Widths for Compact Sets

After the preprarations on local reflexivity, we are now ready to prove the main result, which shows the coincidence of algebraic and continuous n-widths for a large class of sets. Except for the finite-dimensional case, such a result has been known previously only for special sets of functions, where linear and Kolmogorov n-widths were computed exactly and turned out to be equal [15, 17].

3.1. Theorem. *Let X be a normed space, and let $A = K + E$, where K is a compact absolutely convex subset of X and E is a subspace of X of finite dimension m. Then for $n \geqslant m$ the following hold*:

(i) $\lambda_n^{\#}(A, X) = \lambda_n(A, X)$,

(ii) $d^{n,\#}(A, X) = d^n(A, X)$.

Proof. (i) We set $C = K + B_E$. Then C is absolutely convex and compact. Let $X_C = \operatorname{span} C$, and let Y be X_C, endowed with the norm defined by C. Y is a Banach space. Let J: $Y \to X$ be the identical embedding. Then we have $JB_Y = C$. Let $n \geqslant m$, $\varepsilon > 0$, and let $S \in L^{\#}(X)$ be such that $\operatorname{rk} S \leqslant n$ and

$$\sup_{x \in A} \|x - Sx\| \leqslant \lambda_n^{\#}(A, X) + \varepsilon. \tag{10}$$

Since $C \subseteq A$, S must be bounded on C; hence $SJ \in L(Y, X)$ and has a representation

$$SJ = \sum_{k=1}^{n} y_k^* \otimes x_k$$

with $y_k^* \in Y^*$ and $x_k \in X$. Inequality (10) gives

$$\|J^* - (SJ)^*\| = \|J - SJ\| \leqslant \lambda_n^{\#}(A, X) + \varepsilon;$$

thus

$$\sup_{x^* \in B_{X^*}} \left\| J^* x^* - \sum_{k=1}^{n} \langle x_k, x^* \rangle y_k^* \right\| \leqslant \lambda_n^{\#}(A, X) + \varepsilon. \tag{11}$$

Moreover, (10) also implies

$$Sy = y \qquad (y \in E). \tag{12}$$

Next we shall use the local reflexivity 2.2 to produce a bounded linear operator T on X related to S. Since J^* and $(SJ)^*$ are compact operators, there is a finite partition $(\Delta_1, ..., \Delta_M)$ of B_{X^*} into disjoint subsets such that the diameters of the sets $J^*\Delta_j$ and $(SJ)^*\Delta_j$ are not greater than ε for all $1 \leqslant j \leqslant M$. Choose one point u_j from each set Δ_j and let $z_j = J^*u_j$. We define

$$G = \operatorname{span}(\{y_k^* : k = 1, ..., n\} \cup \{z_j : j = 1, ..., M\}).$$

Since $JB_Y = C$ is a compact set, Theorem 2.2 applies and we can find a linear operator $V: G \to Z$ (where, as usual, $Z = J^*X^*$) with $\|V\| \leqslant 1 + \varepsilon$, $Vz_j = z_j$ $(j = 1, ..., M)$, and $\langle y, Vy^* \rangle = \langle y, y^* \rangle$ for all $y \in E$ and $y^* \in G$ (note that $E \subseteq X_C = Y$). Now let $x_k^* \in X^*$ be such that $Vy_k^* = J^*x_k^*$ $(k = 1, ..., n)$. Then we define

$$T = \sum_{k=1}^{n} x_k^* \otimes x_k,$$

$T \in L(X)$, and we get from the above and (12)

$$Ty = y \qquad (y \in E). \tag{13}$$

Finally we verify that T is approximating A sufficiently well. For this, let $x^* \in B_{X^*}$. Then there is a j, $1 \leqslant j \leqslant M$, such that $x^* \in \Delta_j$; hence

$$\|J^*x^* - J^*u_j\| \leqslant \varepsilon$$

and

$$\left\| \sum_{k=1}^{n} \langle x_k, x^* \rangle y_k^* - \sum_{k=1}^{n} \langle x_k, u_j \rangle y_k^* \right\| \leqslant \varepsilon.$$

Using this, we get

$$\begin{aligned}
\|J^*x^* - J^*T^*x^*\| &= \left\| J^*x^* - \sum_{k=1}^{n} \langle x_k, x^* \rangle J^*x_k^* \right\| \\
&\leqslant \left\| z_j - \sum_{k=1}^{n} \langle x_k, x^* \rangle J^*x_k^* \right\| + \varepsilon \\
&= \left\| Vz_j - \sum_{k=1}^{n} \langle x_k, x^* \rangle Vy_k^* \right\| + \varepsilon \\
&\leqslant (1+\varepsilon) \left\| z_j - \sum_{k=1}^{n} \langle x_k, x^* \rangle y_k^* \right\| + \varepsilon \\
&\leqslant (1+\varepsilon) \left\| J^*u_j - \sum_{k=1}^{n} \langle x_k, u_j \rangle y_k^* \right\| + (2+\varepsilon)\varepsilon \\
&\leqslant (1+\varepsilon)\lambda_n^{\#}(A, X) + (3+2\varepsilon)\varepsilon
\end{aligned}$$

(the last inequality by (11)). Using (13), we finally obtain

$$\begin{aligned}\lambda_n(A, X) &\leqslant \sup_{x \in A} \|x - Tx\| \\ &= \sup_{x \in C} \|x - Tx\| \\ &= \|J - TJ\| \\ &= \|J^* - J^* T^*\| \\ &\leqslant (1+\varepsilon)\lambda_n^{\#}(A, X) + (3 + 2\varepsilon)\varepsilon\end{aligned}$$

which yields the result. (ii) follows in the usual way from (i) and Lemma 1.1.

The method of proof of 3.1(i) also applies to n-widths defined by some additional restrictions to the approximating operators. The projectional n-widths are an example [17, 1.1.5]:

$$\pi_n(A, X) = \inf_{\substack{P \in L(X) \\ \operatorname{rk} P \leqslant n,\ P^2 = P}} \sup_{x \in A} \|x - Px\|. \tag{14}$$

Similarly, replacing in (14) $L(X)$ by $L^{\#}(X)$, we define $\pi_n^{\#}(A, X)$. Instead of repeating the proof of 3.1, in this case it is also possible to derive equality directly from the statement of 3.1:

3.2. Corollary. *Let $A = K + E$, where K is an absolutely convex compact subset of a normed space X, and E is a subspace of X of finite dimension m. Then for $n \geqslant m$,*

$$\pi_n^{\#}(A, X) = \pi_n(A, X).$$

Proof. Let $\varepsilon > 0$, $n \geqslant m$, and let $P \in L^{\#}(X)$ be a projection with $\operatorname{rk} P = k \leqslant n$ and

$$\sup_{x \in A} \|x - Px\| \leqslant \pi_n^{\#}(A, X) + \varepsilon.$$

Let $F = PX$. Then $E \subseteq F$ and

$$\lambda_k^{\#}(K + F, X) \leqslant \pi_n^{\#}(A, X) + \varepsilon.$$

By 3.1, $\lambda_k(K + F, X) = \lambda_k^{\#}(K + F, X)$; hence there is a $Q \in L(X)$ with $\operatorname{rk} Q \leqslant k$ and

$$\sup_{x \in K + F} \|x - Qx\| \leqslant \pi_n^{\#}(A, X) + 2\varepsilon.$$

This implies $Qx = x$ for all $x \in F$. Since $\dim F = k = \operatorname{rk} Q$, it follows that Q is a projection, thus

$$\pi_n(A, X) \leqslant \pi_n(K + F, X) \leqslant \pi_n^{\#}(A, X) + 2\varepsilon.$$

In [13, p. 69] Micchelli and Pinkus defined the following generalization of Gelfand width: Let $A \subseteq X$ be an absolutely convex set, let $X_A = \operatorname{span} A$, and let $\mathscr{F} \subseteq X_A^{\#}$ be any set of linear functionals on X_A. Then set

$$d^n(A, X, \mathscr{F}) = \inf_{f_1, \ldots, f_n \in \mathscr{F}} \sup_{\substack{x \in A \\ f_1(x) = \cdots = f_n(x) = 0}} \|x\|.$$

The following is an immediate consequence of 3.1 and generalizes a part of Theorem 3.3 of [13]:

3.3. Corollary. *Let $K \subset X$ be absolutely convex and compact, let $E \subset X$ be a subspace with* $\dim E = m$, *let $A = K + E$, and assume $\mathscr{F} \subseteq X_A^{\#}$ is a subset which contains all restrictions of elements of X^* to X_A. Then for $n \geqslant m$*

$$d^n(A, X, \mathscr{F}) = d^n(A, X).$$

The next corollary directly follows from 3.1 and 1.3. It shows the connection between the widths of a set K and the s-numbers of the related embedding J_K (see the Introduction), and hence allows us to pass results on s-numbers to n-widths.

3.4. Corollary. *Let $K \subset X$ be a compact absolutely convex subset of a normed space X. Then*

$$\lambda_n(K, X) = a_{n+1}(J_K) \qquad \textit{and} \qquad d^n(K, X) = c_{n+1}(J_K).$$

Rephrasing this for embedding operators, we see that $\lambda_n(JB_Y, X) = a_{n+1}(J)$ and $d^n(JB_Y, X) = c_{n+1}(J)$ for each injection $J \in L(Y, X)$ with JB_Y compact. Most of the commonly used embedding operators such as Besov, Sobolev, and Hölder embeddings share this property.

Now let us point out a consequence of 3.1 related to the approximation property of Banach spaces. (The author is grateful to the referee for suggesting this corollary.) A Banach space is said to have the approximation property if for each compact set $K \subset X$ the identity on X can be approximated uniformly on K by elements of $L(X)$ of finite rank [11, I, 1.e.1]. A result of Grothendieck may be phrased as follows: A Banach space has the approximation property iff it satisfies the definition above with $L(X)$ replaced by $L^{\#}(X)$. Although not explicitly stated, this is implied directly by the proof ([2, Prop. 35, $(A_5) \Rightarrow (A)$]; see also [11, I, 1.c.4 (v) $\Rightarrow$ (i)]. To obtain approximation on a given compact set K, the proof uses in an essential way that the assumption involves all absolutely convex compact sets (namely, one includes K into an absolutely convex compact U such that K is compact in X_U, and then uses the hypothesis for U). Theorem 3.1 provides a version of Grothendieck's result for single sets.

3.5. Corollary. *Let K be a compact absolutely convex subset of a Banach space X and assume that there is a sequence of finite rank operators $S_n \in L^{\#}(X)$ such that*

$$\lim_{n \to \infty} \sup_{x \in K} \|x - S_n x\| = 0.$$

Then there is a sequence of continuous finite rank operators $T_n \in L(X)$ with

$$\lim_{n \to \infty} \sup_{x \in K} \|x - T_n x\| = 0.$$

Remarks. 1. The proof of 3.1 gives also some information on the non-compact case: Let $W \subset X$ be an absolutely convex weakly compact set, $E \subset X$ a finite-dimensional subspace, and let $A = W + E$. Then for all $n \geqslant \dim E$,

$$\lambda_n^{\#}(A, X) \leqslant \lambda_n(A, X) \leqslant \lambda_n^{\#}(A, X) + 2 \lim_{k \to \infty} d^{k, \#}(A, X) \tag{15}$$

and

$$d^{n, \#}(A, X) \leqslant d^n(A, X) \leqslant d^{n, \#}(A, X) + 2 \lim_{k \to \infty} d^{k, \#}(A, X). \tag{16}$$

Indeed, the proof goes through with just the following minor changes: Let $\eta = \lim_{k \to \infty} d^{k, \#}(A, X)$. In the terminology of the proof of 3.1, we have by [14, 11.7.6],

$$d_k(J^* B_{X^*}, Y^*) = d_{k+1}(J^*) = c_{k+1}(J) \leqslant d^{k, \#}(A, X).$$

Therefore we can choose the partition $(\Delta_1, ..., \Delta_M)$ in such a way that the diameters of $J^* \Delta_j$ are not greater than $2\eta + \varepsilon$, and those of $(SJ)^* \Delta_j$ are not greater than ε. It follows from (15) and (16) that for $A = W + E$ as above

$$\lambda_n^{\#}(A, X) \leqslant \lambda_n(A, X) \leqslant 3\lambda_n^{\#}(A, X) \tag{17}$$

and

$$d^{n, \#}(A, X) \leqslant d^n(A, X) \leqslant 3d^{n, \#}(A, X). \tag{18}$$

From these relations one can derive the corresponding counterparts to 3.2–3.5.

2. For weakly compact absolutely convex sets, equality between algebraic and continuous widths does not hold, in general. To see this, let $X = l_\infty$, $Y = l_1$, $J: l_1 \to l_\infty$ the identical embedding, $W = JB_Y$. Clearly, W is closed and, since J factors through l_2, W is weakly compact. By 1.3 and [14, 11.11.10], $\lambda_n^\#(W, X) = a_{n+1}(J) = \frac{1}{2}$. On the other hand, we have $Z = J^*X^* \subseteq c_0$, where c_0 is regarded as a subspace of $Y^* = l_\infty$. This together with formula (4) of Section 1 readily implies $\lambda_n(W, X) = 1$.

4. A Counterexample

In this section we give an example which shows that the assumption on compactness in 3.1 and 3.4 cannot be replaced by relative compactness. It shows as well that the estimate in 1.2 is asymptotically sharp. First we need some auxiliary results. Let $I_{1,2}^m: l_1^m \to l_2^m$ be the identity.

4.1. Lemma. (i) $a_n(I_{1,2}^m) = ((m-n+1)/m)^{1/2}$ $(n \leqslant m)$.

(ii) *There is a constant* $\gamma > 0$ *such that for all* $n \leqslant m$

$$\gamma^{-1} c_n(I_{1,2}^m) \leqslant \min\{1, n^{-1/2}(1+\log(m/n))^{1/2}\} \leqslant \gamma c_n(I_{1,2}^m).$$

Here as well as in the sequel we consider logarithms to the basis 2. This is convenient, though not essential, in view of the asymptotic setting. For (i) we refer the reader to [14, 11.11.8]; (ii) is due to Kashin [8, Th. 1] and Garnaev and Gluskin [1]. Let $D: l_1 \to l_2$ be the diagonal mapping defined by $D(\xi_k) = (k^{-1/2}\xi_k)$.

4.2. Lemma. (i) $a_n(D) \asymp n^{-1/2}$.

(ii) $c_n(D) \asymp n^{-1}$.

(We write $\alpha_n \asymp \beta_n$, if there are $\gamma > 0$ and n_0 such that $\gamma^{-1}\beta_n \leqslant \alpha_n \leqslant \gamma\beta_n$ for all $n \geqslant n_0$.) Part (i) and the lower estimate of (ii) are simple consequences of 4.1 and elementary properties of s-numbers [14, 11.1]. The upper estimate is known as well and is a consequence of 4.1(ii) and Maiorov's discretization technique [12]. For completeness, let us indicate the proof of this upper estimate.

Scetch of proof. Let (e_k) be the unit vector basis in l_1, fix a natural m, and choose for each j, $1 \leqslant j \leqslant m$, a subspace E_{m+j} of

$$F_{m+j} = \operatorname{span}\{e_k : 2^{m+j} \leqslant k < 2^{m+j+1}\}$$

of codimension smaller than $2^{m-j}j$ in F_{m+j} such that

$$\|D\,|_{E_{m+j}}\| \leqslant 2^{-(m+j)/2} c_{2^{m-j}j}(I_{1,2}^{2^{m+j}}) \leqslant \gamma_1 2^{-m}$$

for some constant $\gamma_1 > 0$ independent of j and m. Setting

$$G = \operatorname{span}\left(\bigcup_{j=1}^{m} E_{m+j} \cup \{e_k : k \geqslant 2^{2m+1}\} \right)$$

we get

$$\|D\,|_G\| \leqslant \max(\gamma_1, 1) \cdot 2^{-m}$$

and

$$\operatorname{codim} G \leqslant \gamma_2 2^m$$

with $\gamma_2 > 0$ independent of m. This clearly implies that there is a constant $\gamma_3 > 0$ such that for all n,

$$c_n(D) \leqslant \gamma_3 n^{-1}.$$

Now we are ready to give the example.

4.3. THEOREM. *There is an injection $J \in L(l_1, c_0)$ with JB_{l_1} relatively compact such that*

$$a_{n+1}(J) \asymp n^{-1/2} \lambda_n(JB_{l_1}, c_0)$$

and

$$c_{n+1}(J) \asymp n^{-1/2} d^n(JB_{l_1}, c_0).$$

Combining this with 1.3, we obtain

4.4. COROLLARY. *There is an absolutely convex relatively compact subset $K \subset c_0$ (non-trivial in the sense that its span is infinite dimensional) with*

$$\lambda_n^{\#}(K, c_0) \asymp n^{-1/2} \lambda_n(K, c_0)$$

and

$$d^{n,\#}(K, c_0) \asymp n^{-1/2} d^n(K, c_0).$$

Note that the K obtained from 4.3 is even σ-convex, which means that for all sequences (x_k) with $x_k \in K$ and (λ_k) with $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$ we have $\sum_{k=1}^{\infty} \lambda_k x_k \in K$.

Proof of 4.3. Let $Y = l_1$. There is a subspace $H \subset l_\infty = Y^*$ isometric to l_2. H and c_0 are totally incomparable, i.e., no infinite-dimensional subspace of the one space is isomorphic to a subspace of the other [11, I, 2.a.2]. Hence (considering c_0 as a subspace of l_∞) $H \cap c_0$ is finite-dimensional. Let H_0 be the orthogonal complement in H of this intersection. Then $H_0 \cap c_0 = \{0\}$. By a result of Rosenthal [16, Th. 1], $Z = H_0 + c_0$ is closed in l_∞, hence the sum is direct, $Z = H_0 \oplus c_0$. Let (e_k) be the unit vector basis in c_0, (h_k) an orthonormal basis in H_0, and let $X = c_0$ be represented as $X = c_0 \oplus c_0$. (Here and below we do not specify the norms on the direct sums, since we are only interested in asymptotic results.) We define $J\colon Y \to X$ as

$$Jy = (J_1 y, J_2 y)$$

with

$$J_1 y = (k^{-1/2} \langle y, h_k \rangle) \qquad \text{and} \qquad J_2 y = (k^{-1} \langle y, e_k \rangle).$$

JB_Y is relatively compact, and J^*X^* is a dense subspace of Z. Since J_2 is an injection, J is an injection as well. Let $R\colon X^* \to Z$ be J^*, considered as an operator from X^* to Z. Then, identifying H_0 via (h_k) with l_2, it is an easy matter to check that

$$R = D_1 \oplus D_2 \colon l_1 \oplus l_1 \to l_2 \oplus c_0,$$

where $D_1 : l_1 \to l_2$ is as in Lemma 4.2, i.e., $D_1(\xi_k) = (k^{-1/2} \xi_k)$, while $D_2 : l_1 \to c_0$ is defined as $D_2(\xi_k) = (k^{-1} \xi_k)$. By (4) of Section 1,

$$\begin{aligned} \lambda_n(JB_Y, X) &= \inf_{\substack{z_1, \ldots, z_n \in J^*X^* \\ x_1, \ldots, x_n \in X}} \left\| J - \sum_{k=1}^{n} z_k \otimes x_k \right\| \\ &= \inf_{\substack{z_1, \ldots, z_n \in J^*X^* \\ x_1, \ldots, x_n \in X}} \left\| J^* - \sum_{k=1}^{n} x_k \otimes z_k \right\| \\ &\geqslant a_{n+1}(R) \end{aligned}$$

(actually, equality holds, by a local reflexivity argument). By 4.2,

$$a_{n+1}(R) = a_{n+1}(D_1 \oplus D_2) \asymp n^{-1/2}.$$

Using the duality of approximation numbers [14, 11.7.4] and the metric extension property of Y^*, by [14, 11.5.3] we get

$$a_{n+1}(J) = a_{n+1}(J^*) = c_{n+1}(J^*) = c_{n+1}(R)$$

and, by 4.2,

$$c_{n+1}(R)=c_{n+1}(D_1 \oplus D_2) \asymp n^{-1}.$$

Summarizing the above and taking into account 1.2, we get $\lambda_n(JB_Y, X) \asymp n^{1/2}$ and $a_{n+1}(J) \asymp n^{-1}$. This proves the first assertion. To verify the second one, we use the duality of Gelfand and Kolmogorov numbers [14, 11.7.6], the metric lifting property of $X^* = l_1$, and [14, 11.6.3] to get

$$c_{n+1}(J)=d_{n+1}(J^*)=a_{n+1}(J^*);$$

thus by the result above

$$c_{n+1}(J) \asymp n^{-1}.$$

To deal with the Gelfand width, we first apply a separation argument which gives for $x^*, x_1^*, ..., x_n^* \in X^*$

$$\begin{aligned}&\sup\{|\langle y, J^*x^*\rangle| : y \in B_Y, \langle y, J^*x_1^*\rangle = \cdots = \langle y, J^*x_n^*\rangle = 0\}\\ &\quad = \inf\{\|J^*x^* - u\| : u \in \operatorname{span}(J^*x_1^*, ..., J^*x_n^*)\}.\end{aligned}$$

Therefore,

$$\begin{aligned} d^n(JB_Y, X) &= \inf_{x_1^*, ..., x_n^* \in X^*} \sup_{x^* \in B_{X^*}} \\ &\quad \times \sup\{|\langle Jy, x^*\rangle| : y \in B_Y, \langle Jy, x_1^*\rangle = \cdots = \langle Jy, x_n^*\rangle = 0\} \\ &= \inf_{x_1^*, ..., x_n^*} \sup_{x^* \in B_{X^*}} \inf\{\|J^*x^* - u\| : u \in \operatorname{span}(J^*x_1^*, ..., J^*x_n^*)\} \\ &= d_n(J^*B_{X^*}, J^*X^*) \\ &= d_{n+1}(R) \end{aligned}$$

(compare also [17, 2.6.5]). Again by the metric lifting property of X^* and by the above

$$d_{n+1}(R) = a_{n+1}(R) \asymp n^{-1/2},$$

thus

$$d^n(JB_Y, X) \asymp n^{-1/2}.$$

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